Estimates for fractional integral operator and its commutators on grand p-adic Herz-Morrey spaces

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Abstract: In this article, the main aim is to introduce the grand variable Herz space over the p-adic fields and demonstrate the boundedness for fractional integral operator, fractional maximal operator in the context of the grand p-adic version of Herz-Morrey spaces with variable exponent, as well as the Lipschitz estimates for the commutators of fractional integral operator, fractional maximal operator, and sharp maximal function on the grand p-adic version of Herz-Morrey spaces with variable exponent.

Keywords: p-adic field; fractional integral operator; grand space; variable exponent Herz-Morrey space; commutator

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1 Introduction and main resulats

In the previous decades, the development of p-adic analysis is becoming more and more rapid and rich, which depends on its powerful applications. Among them, the application in physics focuses on the theory of p-adic strings and complex disordered systems-spin glasses, and quantum mechanics [1, 2, 19]. Besides, it also has substantial important applications in biology and geology, exactly speaking, mathematical methods on p-adic analysis can ofen reveal some biological phenomena [4, 11], and the theory of p-adic analysis can also deal with fractal problem in geology [6, 7].

We now first introduce some fundamental notions on p-adic field. Let $\mathbb{Z}$ be the field of integer. For a fixed prime number p, the p-adic field $\mathbb{Q}_p$, which originally given by K. Hensel in 1897, is composed of the rational numbers field $\mathbb{Q}$ with respect to non-Archimedean p-adic absolute value: let $x = p^{\gamma}\frac{a}{b}$, where $x \in \mathbb{Q}$ and $\gamma \in \mathbb{Z}$, a and b are non-zero integers which are not divisible by p, the p-adic absolute value is $|x|_p = p^{-\gamma}$.

It is well known that the non-Archimedean p-adic absolute value has many properties similar to the Archimedean absolute value, for instance, positive definiteness, product properties and

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non-Archimedean p-adic absolute value inequality. Exactly speaking, these properties are shown as follows.
(1) $|x|_p \geq 0$. Specially, $|x|_p = 0$ if and only if $x = 0$;
(2) $|xy|_p = |x|_p|y|_p$;
(3) $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. If $|x|_p \neq |y|_p$. then the equality holds and the converse is also true.

Combining Properties (1) and (3), we also obtain the same triangle inequality as Archimedean absolute value, namely, $|x+y|_p \leq |x|_p + |y|_p$.

From the standard p-adic analysis, nonzero p-adic number x can be written as:

$$x = p^{\gamma}(a_0 + a_1 p + a_2 p^2 + \cdots) = p^{\gamma} \sum_{j=0}^{\infty} a_j p^j, \quad a_j = 0, \cdots, p-1,$$

where $|x|_p = p^{-\gamma}$ when $a_\gamma \neq 0$. Naturally, the above p-adic number x converges.

In the next time, we need to further consider the n-dimensional p-adic linear space $\mathbb{Q}_p^n$, when $n = 1$, this case is shown in the description above. For any n-dimensional vector $x = (x_1, x_2, \cdot, \cdot, \cdot, x_n)$, where $x_i \in \mathbb{Q}_p$ $(i = 1, ..., n)$, then the following p-adic absolute value is given by

$$|x|_p = \max_{1 \leq j \leq n} |x_j|_p.$$

Finally, the p-adic ball is denoted by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\},$$

where the center of p-adic ball $a \in \mathbb{Q}_p^n$ and radius p^γ with $\gamma \in \mathbb{Z}$. The p-adic corresponding sphere is denoted by

$$S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\} = B_\gamma(a) \setminus B_{\gamma-1}(a).$$

Specially, if $a, \gamma = 0$, then the $B_0(0)$ and $S_0(0)$ are called the p-adic unit ball and p-adic unit sphere, respectively.

Moreover, when $a = 0$, we usually omit the center of p-adic ball and sphere. From the definition of p-adic ball and sphere, we observe a relation between them, exactly speaking, $B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a)$ and $\mathbb{Q}_p^n \setminus \{0\} = \bigcup_{\gamma \in \mathbb{Z}} S_\gamma$. For any $a_0 \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, it is not difficult to obtain the following equalities

$$a_0 + B_\gamma = B_\gamma(a_0) \text{ and } a_0 + S_\gamma = S_\gamma(a_0) = B_\gamma(a_0) \setminus B_{\gamma-1}(a_0).$$

For the sake of simplicity, we define the characteristic function $\chi_k = \chi_{S_k} = \chi_{B_k \setminus B_{k-1}}$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists Haar measure on $\mathbb{Q}_p^n$, it is easy to know that unique Haar measure $\mathrm{d}x$ on $\mathbb{Q}_p^n$ (up to positive constant multiple)

are translation invariant (i.e., $\mathrm{d}(x+a)=\mathrm{d}x$). Then we integrate on p-adic unit ball firstly, such that

$$\int_{B_0} dx = |B_0|_h = 1,$$

where $|B_0|_h$ is denoted by the Haar measure of p-adic unit ball. Generally speaking, for any $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, we have

$$\int_{B_\gamma(a)} dx = |B_\gamma(a)|_h = p^{n\gamma}$$

and

$$\int_{S_\gamma(a)} dx = |S_\gamma(a)|_h = p^{n\gamma}(1-p^{-n}) = |B_\gamma(a)|_h - |B_{\gamma-1}(a)|_h.$$

For more details about the p-adic analysis, we refer readers to [18, 19] and references therein.

It is well-known that the study of operator theory has caught a lot of attention due to many applications in partial differential equations and harmonic analysis, where the main concern is its boundedness in different spaces. In this article, the studies are oriented to the following p-adic fractional integral operator.

Let $0<\alpha<n$, we define the p-adic fractional integral operator as

$$I_\alpha^p(f)(x) = \int_{\mathbb{Q}_p^n} \frac{f(y)}{|x-y|_p^{n-\alpha}} dy.$$

As a vast branch of harmonic analysis, the space of variable functions is a generalization of some classical function spaces, such as the variable exponent Lebesgue space is a generalization of the classical Lebesgue space, and the Herz-Morrey space is a generalization of the Herz space. On the one hand, Cortés and Rafeiro [8] introduced p-adic version of variable exponent Lebesgue spaces, and obtained many properties and application. The boundedness of the fractional integral operator and fractional maximal operator was obtained in [10]. On the other hand, Sarfraz, Aslam, Zaman, and Jarad [13] obtained the estimates for fractional integral operator on the p-adic Herz-Morrey space. Recently, the grand function space with variable exponent have a positive development trend. In the Euclidean spaces, the boundedness of fractional integral operator on grand Herz-Morrey spaces was given in [14]. Sultan et al. [15] defined the grand p-adic Herz-Morrey spaces with variable exponent and obtained the boundedness of an intrinsic square function. Therefore, the study of p-adic grand Herz-Morrey spaces with variable exponent in p-adic linear spaces is quite a few, which look worthy of further investigations.

Inspired by the aforementioned literature, the key consideration is p-adic fields $\mathbb{Q}_p$, in a few cases it can be also indicated that our work is motivated by the standard harmonic analysis

on the Euclidean space. The purpose of this paper is to study the boundedness for fractional integral operator and fractional maximal operator in the context of the p-adic version of Herz-Morrey spaces with variable exponent, as well as the Lipschitz estimates for the commutators of fractional integral operator, fractional maximal operator and sharp maximal function on the grand p-adic version of Herz-Morrey spaces with variable exponent.

Naturally, we first need to consider the follwing result of boundedness of the fractional integral operator I^p_α on the p-adic vector spaces.

Theorem 1.1 Let $1 \leq u < \infty,\ 0 < \lambda < \infty,\ 0 < \alpha < n$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$, with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\alpha,$ and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\lambda + \alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\lambda + \alpha - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then I^p_α is bounded from $MK^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)$ to $MK^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{Q}_p^n)$.

Remark 1 For the results of the aforementioned case in Euclidean space, see [14].

On the one hand, when $\eta(\cdot),\ q_1(\cdot),$ and $q_2(\cdot)$ in theorem 1.1 are both constant exponent, this results are still new.

Corollary 1.1 Let $1 \leq u < \infty,\ \ 0 < \lambda < \infty,\ 0 < \alpha < n$, for all $q_1,\ q_2$ with $1 < q_1 < n/\alpha$ and $1/q_2 = 1/q_1 - \alpha/n$. If $\alpha + \lambda - n/q_1 < \eta < n/q_1'$. Then I^p_α is bounded from $MK^{\eta,u),\theta}_{\lambda,q_1}(\mathbb{Q}_p^n)$ to $MK^{\eta,u),\theta}_{\lambda,q_2}(\mathbb{Q}_p^n)$.

For the case of $\lambda = 0$ in the corollary 1.1, it reveals the boundedness of p-adic fractional integral operator on grand Herz spaces.

Corollary 1.2 Let $1 \leq u < \infty,\ 0 < \alpha < n,$, for all $q_1,\ q_2$ with $1 < q_1 < n/\alpha$ and $1/q_2 = 1/q_1 - \alpha/n$. If $\ \alpha - n/q_1 < \eta < n/q_1'$. Then I^p_α is bounded from $K^{\eta,u),\theta}_{q_1}(\mathbb{Q}_p^n)$ to $K^{\eta,u),\theta}_{q_2}(\mathbb{Q}_p^n)$.

On the other hand, for the case of $\lambda = 0$ in the theorem 1.1, the boundedness of the fractional integral operator on grand p-adic variable Herz space is obtained and it is even new.

Corollary 1.3 Let $1 \leq u < \infty,\ 0 < \alpha < n$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\alpha,$ and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\alpha - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then I^p_α is bounded from $K^{\eta(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{Q}_p^n)$ to $K^{\eta(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{Q}_p^n)$.

For the case of $\eta(\cdot)$ be a constant exponent, this result is also even new in p-adic vector space.

Corollary 1.4 Let $1 \leq u < \infty,\ 0 < \alpha < n$, for all $q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\alpha,$ and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If η is satisfying the following conditions

(1) $\alpha - n/q_1(\cdot) < \eta < n/q_1'(\cdot)$,

(2) $\alpha - n/q_1(\infty) < \eta < n/q_1'(\infty)$.

Then I_α^p is bounded from $K_{q_1(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

Let $0 < \alpha < n$, we define the p-adic fractional maximal operator as

$$M_\alpha^p(f)(x) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}} \int_{B_\gamma(x)} |f(y)|dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$.

Then the maximal commutator of M_α^p with b is given by

$$M_{\alpha,b}^p(f)(x) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}} \int_{B_\gamma(x)} |b(x) - b(y)||f(y)|dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$.

Moreover, assume that $b : \mathbb{Q}_p^n \to \mathbb{R}$ and $f : \mathbb{Q}_p^n \to \mathbb{R}$ are measurable mappings, then the nonlinear commutators of fractional maximal operator can be defined as follows:

$$[b, M_\alpha^p]f(x) = b(x)M_\alpha^p(f)(x) - M_\alpha^p(bf)(x).$$

If $\alpha = 0$, we have $[b, M^p] = [b, M_0^p]$ and $M_b^p = M_{0,b}^p$.

Inspired by Sobolev inequality, the boundedness of the following fractional maximal operator on the grand p-adic Herz-Morrey space with variable exponent is given.

Theorem 1.2 Let $1 \leq u < \infty,\ 0 < \lambda < \infty,\ 0 < \alpha < n$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\alpha,$ and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\alpha + \lambda - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\alpha + \lambda - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

Then M_α^p is bounded from $MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

On the one hand, when $\eta(\cdot),\ q_1(\cdot),$ and $q_2(\cdot)$ in theorem 1.2 are both constant exponent, this results are still new.

Corollary 1.5 Let $1 \leq u < \infty,\ 0 < \lambda < \infty,\ 0 < \alpha < n$, for all $q_1,\ q_2$ with $1 < q_1 < n/\alpha$ and $1/q_2 = 1/q_1 - \alpha/n$. If $\alpha + \lambda - n/q_1 < \eta < n/q_1'$. Then M_α^p is bounded from $MK_{\lambda,q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

For the case of $\lambda = 0$ in the corollary 1.5, it reveals the boundedness of p-adic fractional maximal operator on grand Herz spaces.

Corollary 1.6 Let $1 \leq u < \infty,\ 0 < \alpha < n$, for all $q_1,\ q_2$ with $1 < q_1 < n/\alpha$ and $1/q_2 = 1/q_1 - \alpha/n$. If $\alpha - n/q_1 < \eta < n/q_1'$. Then M_α^p is bounded from $K_{q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

On the other hand, for the case of $\lambda = 0$ in the theorem 1.2, the boundedness of the fractional maximal operator on grand p-adic variable Herz space is obtained and it is even new.

Corollary 1.7 Let $1 \leq u < \infty,\ 0 < \alpha < n$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\alpha,$ and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\alpha - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$,

Then M_α^p is bounded from $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

For the case of $\eta(\cdot)$ be a constant exponent, this result is also even new in p-adic vector space.

Corollary 1.8 Let $1 \leq u < \infty,\ 0 < \alpha < n$, for all $q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\alpha,$ and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$. If η is satisfying the following conditions

(1) $\alpha - n/q_1(\cdot) < \eta < n/q_1'(\cdot)$,

(2) $\alpha - n/q_1(\infty) < \eta < n/q_1'(\infty)$,

and $0 < \eta$. Then M_α^p is bounded from $K_{q_1(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

Furthermore, assume that $b:\ \mathbb{Q}_p^n \to \mathbb{R}$ and $f:\ \mathbb{Q}_p^n \to \mathbb{R}$ are measurable mappings, then the commutators of fractional integral operator can be defined as follows:

$$[b, I_\alpha^p](f)(x) = b(x)I_\alpha^p f(x) - I_\alpha^p(bf)(x) = \int_{\mathbb{Q}_p^n} \frac{(b(x) - b(y))f(y)}{|x-y|_p^{n-\alpha}} dy. \tag{1.1}$$

Now, we give the Lipschitz estimates for the (nonlinear) commutator of fractional integral operator and fractional maximal operator on the grand p-adic Herz-Morrey space with variable exponent.

Theorem 1.3 Let $1 \leq u < \infty,\ 0 < \lambda < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/(\alpha+\beta),$ and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\lambda + \alpha + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\lambda + \alpha + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then $[b, I_\alpha^p]$ is bounded from $MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

On the one hand, when $\eta(\cdot),\ q_1(\cdot),$ and $q_2(\cdot)$ in theorem 1.3 are both constant exponent, this results are still new.

Corollary 1.9 Let $1 \leq u < \infty,\ 0 < \lambda < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $q_1,\ q_2$ with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$. If $\lambda + \alpha + \beta - n/q_1 < \eta < n/q_1'$. Then $[b, I_\alpha^p]$ is bounded from $MK_{\lambda,q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

For the case of $\lambda = 0$ in the corollary 1.9, it reveals the boundedness from the commutator of p-adic fractional integral operator on grand Herz spaces.

Corollary 1.10 Let $1 \leq u < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $q_1,\ q_2$ with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$. If $\alpha + \beta - n/q_1 < \eta < n/q_1'$. Then $[b, I_\alpha^p]$ is bounded from $K_{q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

On the other hand, for the case of $\lambda = 0$ in the theorem 1.3, the boundedness for the commutator of fractional integral operator on grand p-adic variable Herz space is obtained and it is even new.

Corollary 1.11 Let $1 \leq u < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/(\alpha+\beta),$ and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\alpha + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\alpha + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then $[b, I_\alpha^p]$ is bounded from $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

For the case of $\eta(\cdot)$ be a constant exponent, this result is also even new in p-adic vector space.

Corollary 1.12 Let $1 \leq u < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/(\alpha+\beta),$ and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$. If η is satisfying the following conditions

(1) $\alpha + \beta - n/q_1(\cdot) < \eta < n/q_1'(\cdot)$,

(2) $\alpha + \beta - n/q_1(\infty) < \eta < n/q_1'(\infty)$.

Then $[b, I_\alpha^p]$ is bounded from $K_{q_1(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

According to the aforementioned theorem, naturally, we give the following result and it is even new.

Theorem 1.4 Let $1 \leq u < \infty,\ 0 < \lambda < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/(\alpha+\beta),$ and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\lambda + \alpha + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\lambda + \alpha + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then $M_{\alpha,b}^p$ is bounded from $MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

On the one hand, when $\eta(\cdot),\ q_1(\cdot),$ and $q_2(\cdot)$ in theorem 1.4 are both constant exponent, this results are still new.

Corollary 1.13 Let $1 \le u < \infty,\ \ 0 < \lambda < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $q_1,\ q_2$ with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$. If $\lambda+\alpha+\beta-n/q_1 < \eta < n/q_1'$. Then $M_{\alpha,b}^p$ is bounded from $MK_{\lambda,q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

For the case of $\lambda = 0$ in the corollary 1.13, it reveals the boundedness for commutator of p-adic fractional maximal operator on grand Herz spaces.

Corollary 1.14 Let $1 \le u < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $q_1,\ q_2$ with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n$. If $\alpha+\beta-n/q_1 < \eta < n/q_1'$. Then $M_{\alpha,b}^p$ is bounded from $K_{q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

On the other hand, for the case of $\lambda = 0$ in the theorem 1.4, the boundedness for the commutator of fractional maximal operator on grand p-adic variable Herz space is obtained and it is even new.

Corollary 1.15 Let $1 \le u < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/(\alpha+\beta),$ and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\alpha+\beta-n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\alpha+\beta-n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then $M_{\alpha,b}^p$ is bounded from $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

For the case of $\eta(\cdot)$ be a constant exponent, this result is also even new in p-adic vector space.

Corollary 1.16 Let $1 \le u < \infty,\ 0 < \alpha < \alpha + \beta < n,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/(\alpha+\beta),$ and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$. If η is satisfying the following conditions

(1) $\alpha+\beta-n/q_1(\cdot) < \eta < n/q_1'(\cdot)$,

(2) $\alpha+\beta-n/q_1(\infty) < \eta < n/q_1'(\infty)$.

Then $M_{\alpha,b}^p$ is bounded from $K_{q_1(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

For the case of $\alpha = 0$, the following are obtained by theorem 1.4 and it is even new.

Corollary 1.17 Let $1 \le u < \infty,\ 0 < \lambda < \infty,$ and $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\beta,$ and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\lambda+\beta-n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\lambda+\beta-n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then M_b^p is bounded from $MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

Theorem 1.5 Let $1 \leq u < \infty,\ 0 < \lambda < \infty,\ 0 < \alpha < \alpha + \beta < n,\ b \in \Lambda_\beta(\mathbb{Q}_p^n)\ (0 < \beta < 1),$ and $b \geq 0,$ for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/(\alpha+\beta),$ and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n.$ If $\eta(\cdot)$ is satisfying the following conditions

(1) $\lambda + \alpha + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot),$

(2) $\lambda + \alpha + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty).$

Then $[b, M_\alpha^p]$ is bounded from $MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n).$

On the one hand, when $\eta(\cdot),\ q_1(\cdot),$ and $q_2(\cdot)$ in theorem 1.5 are both constant exponent, this results are still new.

Corollary 1.18 Let $1 \leq u < \infty,\ 0 < \lambda < \infty,\ 0 < \alpha < \alpha + \beta < n,\ b \in \Lambda_\beta(\mathbb{Q}_p^n)\ (0 < \beta < 1),$ and $b \geq 0,$ for all $q_1,\ q_2$ with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n.$ If $\lambda + \alpha + \beta - n/q_1 < \eta < n/q_1'.$ Then $[b, M_\alpha^p]$ is bounded from $MK_{\lambda,q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n).$

For the case of $\lambda = 0$ in the corollary 1.18, it reveals the boundedness for nonlinear commutator of p-adic fractional maximal operator on grand Herz spaces.

Corollary 1.19 Let $1 \leq u < \infty,\ 0 < \alpha < \alpha+\beta < n,\ b \in \Lambda_\beta(\mathbb{Q}_p^n)\ (0 < \beta < 1),$ and $b \geq 0,$ for all $q_1,\ q_2$ with $1 < q_1 < n/(\alpha+\beta)$ and $1/q_2 = 1/q_1 - (\alpha+\beta)/n.$ If $\alpha + \beta - n/q_1 < \eta < n/q_1'.$ Then $[b, M_\alpha^p]$ is bounded from $K_{q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n).$

On the other hand, for the case of $\lambda = 0$ in the theorem 1.5, the boundedness for the nonlinear commutator of fractional maximal operator on grand p-adic variable Herz space is obtained and it is even new.

Corollary 1.20 Let $1 \leq u < \infty,\ 0 < \alpha < \alpha+\beta < n,\ b \in \Lambda_\beta(\mathbb{Q}_p^n)\ (0 < \beta < 1),$ and $b \geq 0,$ for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/(\alpha+\beta),$ and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n.$ If $\eta(\cdot)$ is satisfying the following conditions

(1) $\alpha + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot),$

(2) $\alpha + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty).$

Then $[b, M_\alpha^p]$ is bounded from $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n).$

For the case of $\eta(\cdot)$ be a constant exponent, this result is also even new in p-adic vector space.

Corollary 1.21 Let $1 \leq u < \infty,\ 0 < \alpha < \alpha+\beta < n,\ b \in \Lambda_\beta(\mathbb{Q}_p^n)\ (0 < \beta < 1),$ and $b \geq 0,$ for all $q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/(\alpha+\beta),$ and $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n.$ If η is satisfying the following conditions

(1) $\alpha + \beta - n/q_1(\cdot) < \eta < n/q_1'(\cdot),$

(2) $\alpha + \beta - n/q_1(\infty) < \eta < n/q_1'(\infty).$

Then $[b, M_\alpha^p]$ is bounded from $K_{q_1(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n).$

For the case of $\alpha = 0$, the following are obtained by theorem 1.5 and it is even new.

Corollary 1.22 Let $1 \le u < \infty,\ 0 < \lambda < \infty,\ b \in \Lambda_\beta(\mathbb{Q}_p^n)\ (0 < \beta < 1),$ and $b \ge 0$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\beta,$ and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\lambda + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\lambda + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then $[b, M^p]$ is bounded from $MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

In order to introduce the following theorem, we firstly need to introduce the p-adic version of sharp maximal function $M_p^\sharp$, for a locally integrable function f on $\mathbb{Q}_p^n$, then, in[3], define that

$$M_p^\sharp(f)(x) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$ and $f_{B_\gamma(x)} = \frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)} f(y)dy$.

The commutator generated by $M_p^\sharp$ and $b \in L_{\text{loc}}(\mathbb{Q}_p^n)$ is given by

$$[b, M_p^\sharp](f)(x) = b(x)M_p^\sharp(f)(x) - M_p^\sharp(bf)(x).$$

Theorem 1.6 Let $1 \le u < \infty,\ 0 < \lambda < \infty,\ b \in \Lambda_\beta(\mathbb{Q}_p^n)\ (0 < \beta < 1),$ and $b \ge 0$, for all $\eta(\cdot),\ q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\beta,$ and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$, If $\eta(\cdot)$ is satisfying the following conditions

(1) $\lambda + \beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\lambda + \beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then $[b, M_p^\sharp]$ is bounded from $MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

On the one hand, when $\eta(\cdot),\ q_1(\cdot),$ and $q_2(\cdot)$ in theorem 1.6 are both constant exponent, this results are still new.

Corollary 1.23 Let $1 \le u < \infty,\ 0 < \lambda < \infty,\ b \in \Lambda_\beta(\mathbb{Q}_p^n)\ (0 < \beta < 1),$ and $b \ge 0$, for all $q_1,\ q_2$ with $1 < q_1 < n/\beta$ and $1/q_2 = 1/q_1 - \beta/n$. If $\lambda + \beta - n/q_1 < \eta < n/q_1'$. Then $[b, M_p^\sharp]$ is bounded from $MK_{\lambda,q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $MK_{\lambda,q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

For the case of $\lambda = 0$ in the corollary 1.23, it reveals the boundedness for nonlinear commutator of p-adic sharp maximal function on grand Herz spaces.

Corollary 1.24 Let $1 \le u < \infty,\ b \in \Lambda_\beta(\mathbb{Q}_p^n)\ (0 < \beta < 1),$ and $b \ge 0$, for all $q_1,\ q_2$ with $1 < q_1 < n/\beta$ and $1/q_2 = 1/q_1 - \beta/n$. If $\beta - n/q_1 < \eta < n/q_1'$. Then $[b, M_p^\sharp]$ is bounded from $K_{q_1}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

On the other hand, for the case of $\lambda = 0$ in the theorem 1.5, the boundedness for the commutator of sharp maximal function on grand p-adic variable Herz space is obtained and it is even new.

Corollary 1.25 Let $1 \leq u < \infty$, $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, and $b \geq 0$, for all $\eta(\cdot)$, $q_1(\cdot)$, $q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\beta$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$. If $\eta(\cdot)$ is satisfying the following conditions

(1) $\beta - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$,

(2) $\beta - n/q_1(\infty) < \eta(\cdot) < n/q_1'(\infty)$.

Then $[b, M_p^\sharp]$ is bounded from $K_{q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$.

For the case of $\eta(\cdot)$ be a constant exponent, this result is also even new in p-adic vector space.

Corollary 1.26 Let $1 \leq u < \infty$, $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ $(0 < \beta < 1)$, and $b \geq 0$, for all $q_1(\cdot)$, $q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $(q_1)_+ < n/\beta$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \beta/n$. If η is satisfying the following conditions

(1) $\beta - n/q_1(\cdot) < \eta < n/q_1'(\cdot)$,

(2) $\beta - n/q_1(\infty) < \eta < n/q_1'(\infty)$.

Then $[b, M_p^\sharp]$ is bounded from $K_{q_1(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$ to $K_{q_2(\cdot)}^{\eta,u),\theta}(\mathbb{Q}_p^n)$.

Throughout this paper, the letter C always takes place of a constant independent of the primary parameters involved and whose value may differ from line to line. In addition, we give some notations. Here and hereafter $|E|_h$ will always denote the Haar measure of a measurable set E on $\mathbb{Q}_p^n$ and by χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{Q}_p^n$.

2 Preliminaries

2.1 p-adic function spaces

Assume that $1 \leq q < \infty$, we denote $L^q(\mathbb{Q}_p^n)$ as the p-adic Lebesgue space, the space of all functions f is in the locally L^q space with finite norm

$$\|f\|_{L^q(\mathbb{Q}_p^n)} = (\int_{\mathbb{Q}_p^n} |f(x)|^q dx)^{\frac{1}{q}}.$$

In addition, for $q = \infty$ and denote $L^\infty(\mathbb{Q}_p^n)$ as the set of all measurable real-valued functions f on $\mathbb{Q}_p^n$ satisfying

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)} = \operatorname*{ess\,sup}_{x \in \mathbb{Q}_p^n} |f(x)| = \inf\{\lambda > 0 : |x \in \mathbb{Q}_p^n : |f(x)| > \lambda|_h\} < \infty.$$

Here, if the limit exists, the integral in above equation is defined as follows:

$$\int_{\mathbb{Q}_p^n} |f(x)|^q dx = \lim_{\gamma\to\infty} \int_{B_\gamma(0)} |f(x)|^q dx = \lim_{\gamma\to\infty} \sum_{-\infty<k\leq\gamma} \int_{S_k(0)} |f(x)|^q dx.$$

In particular, since $\mathbb{Q}_p^n = \bigcup_{\gamma=-\infty}^{+\infty} S_\gamma$, and $\mathrm{d}(tx) = |t|_p^n \mathrm{d}x (t \in \mathbb{Q}_p \setminus \{0\})$, if $f \in L^1(\mathbb{Q}_p^n)$, then

$$\int_{\mathbb{Q}_p^n} f(x)dx = \sum_{\gamma=-\infty}^{+\infty} \int_{S_\gamma} f(x)dx$$

and

$$\int_{\mathbb{Q}_p^n} f(tx)dx = \frac{1}{|t|_p^n} \int_{\mathbb{Q}_p^n} f(x)dx.$$

We say that a measurable function $q(\cdot)$ is a variable exponent if $q(\cdot) : \mathbb{Q}_p^n \to (0.\infty)$, the following definition give some notations on the p-adic variable exponent Lebesgue space, which are derived from [17].

Definition 2.1 Given a measurable function $q(\cdot)$ defined on $\mathbb{Q}_p^n$, we denote by

$$q_- := \operatorname*{ess\,inf}_{x\in\mathbb{Q}_p^n} q(x), \quad q_+ := \operatorname*{ess\,sup}_{x\in\mathbb{Q}_p^n} q(x).$$

(1) $q'_- = \operatorname{ess\,inf}_{x\in\mathbb{Q}_p^n} q'(x) = \frac{q_+}{q_+-1}$, $q'_+ = \operatorname{ess\,inf}_{x\in\mathbb{Q}_p^n} q'(x) = \frac{q_-}{q_--1}$.
(2) Denote by $\mathscr{P}(\mathbb{Q}_p^n)$ the set of all measurable function $q(\cdot) : \mathbb{Q}_p^n \to (1, \infty)$ such that

$$1 \leq q_- \leq q(x) \leq q_+ < \infty, \quad x \in \mathbb{Q}_p^n.$$

Definition 2.2 (**p-adic variable exponent Lebesgue spaces**) Let $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$. Define the p-adic variable exponent Lebesgue spaces $L^{q(\cdot)}(\mathbb{Q}_p^n)$ as follows

$$L^{q(\cdot)}(\mathbb{Q}_p^n) = \{f \text{ is measurable function} : \mathscr{F}_q\left(f/\eta\right) < \infty \text{ for some constant } \eta > 0\},$$

where

$$\mathscr{F}_q(f) := \int_{\mathbb{Q}_p^n} |f(x)|^q(x) dx.$$

The Lebesgue space $L^{q(\cdot)}(\mathbb{Q}_p^n)$ is a Banach function space with respect to the Luxemburg norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} = \inf\{\eta > 0 : \mathscr{F}_q(f/\eta) = \int_{\mathbb{Q}_p^n} \left(\frac{|f(x)|}{\eta}\right)^{q(x)} dx \leq 1\}.$$

Next, Cortés and Rafeiro [8] introduced the following class of exponents.

Definition 2.3 (log $-$Hölder **continuity**) Let measurable function $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$.

(1) Denote by $\mathscr{C}_0^{\log}(\mathbb{Q}_p^n)$ the set of all $q(\cdot)$ which satisfies

$$\gamma(q_-(B_\gamma(x)) - q_+(B_\gamma(x))) \leq C,$$

for all $\gamma \in \mathbb{Z}$ and any $x \in \mathbb{Q}_p^n$, where C denotes a universal constant.

(2) The set $\mathscr{C}_\infty^{\log}(\mathbb{Q}_p^n)$ consists of all $q(\cdot)$ which satisfies

$$|q(x) - q(y)| \leq \frac{C}{\log_p(p + \min\{|x|_p, |y|_p\})},$$

for any $x, y \in \mathbb{Q}_p^n$, where C denotes a universal constant.

(3) (see [17]) Denote by $\mathscr{C}^{\log}(\mathbb{Q}_p^n) = \mathscr{C}_0^{\log}(\mathbb{Q}_p^n) \cap \mathscr{C}_\infty^{\log}(\mathbb{Q}_p^n)$ the set of all global log $-$Hölder continuous functions $q(\cdot)$.

Finally, we will introduce p-adic Herz-Morrey spaces with variable exponent and grand p-adic Herz-Morrey spaces with variable exponent [5, 15].

Definition 2.4 Assume $\eta(\cdot) \in L^\infty(\mathbb{Q}_p^n)$, $1 \leq u < \infty$, $s(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $\theta > 0$ and $0 \leq \lambda < \infty$. A homogeneous p-adic Herz-Morrey spaces with variable exponent $MK_{u,s(\cdot)}^{\eta,\lambda}(\mathbb{Q}_p^n)$ can be defined by

$$MK_{u,s(\cdot)}^{\eta(\cdot),\lambda}(\mathbb{Q}_p^n) = \{g \in L_{\text{loc}}^{s(\cdot)}(\mathbb{Q}_p^n \setminus \{0\}) : \|g\|_{MK_{\lambda,s(\cdot)}^{\eta,u}(\mathbb{Q}_p^n)} < \infty\},$$

where the norm

$$\|g\|_{MK_{u,s(\cdot)}^{\eta(\cdot),\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda}\{\sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku}\|g\chi_k\|_{L^{s(\cdot)}}^u\}^{\frac{1}{u}}.$$

Definition 2.5 Assume $\eta(\cdot) \in L^\infty(\mathbb{Q}_p^n)$, $1 \leq u < \infty$, $s(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $\theta > 0$ and $0 \leq \lambda < \infty$. A homogeneous grand p-adic Herz-Morrey spaces with variable exponent $MK_{\lambda,s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ can be defined by

$$MK_{\lambda,s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n) = \{g \in L_{\text{loc}}^{s(\cdot)}(\mathbb{Q}_p^n \setminus \{0\}) : \|g\|_{MK_{\lambda,s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)} < \infty\},$$

where the norm

$$\|g\|_{MK_{\lambda,s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)} = \sup_{\substack{k_0 \in \mathbb{Z} \\ \epsilon > 0}} p^{-k_0\lambda}\{\epsilon^\theta \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\|g\chi_k\|_{L^{s(\cdot)}}^{u(1+\epsilon)}\}^{\frac{1}{u(1+\epsilon)}}.$$

Assume $\lambda = 0$, the grand Herz-Morrey spaces $MK_{\lambda,s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n) = K_{s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$, then the grand p-adic variable Herz space is displayed as follows, the definition in the context of Euclidean space can be found in [9].

Definition 2.6 Assume $\eta(\cdot) \in L^{\infty}(\mathbb{Q}_p^n),\ 1 \leq u < \infty,\ s(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ \theta > 0$. The homogeneous grand p-adic Herz space with variable exponent $K_{s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$ can be defined by

$$K_{s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n) = \{g \in L^{s(\cdot)}(\mathbb{Q}_p^n) : \|g\|_{K_{s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)} < \infty\},$$

where the norm

$$\|g\|_{K_{s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)} = \sup_{\substack{k_0\in\mathbb{Z} \\ \epsilon>0}} \{\epsilon^{\theta} \sum_{k=-\infty}^{\infty} p^{\eta(\cdot)ku(1+\epsilon)} \|g\chi_k\|_{L^{s(\cdot)}}^{u(1+\epsilon)}\}^{\frac{1}{u(1+\epsilon)}}.$$

The following result introduce the basic definition of p-adic Lipschitz spaces [5].

Definition 2.7 Assume $0<\beta<1$. then the p-adic version of homogeneous Lipschitz spaces $\Lambda_{\beta}(\mathbb{Q}_p^n)$ is defined by

$$\Lambda_{\beta}(\mathbb{Q}_p^n) := \{f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{Q}_p^n) :\ \|f\|_{\Lambda_{\beta}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\Lambda_{\beta}(\mathbb{Q}_p^n)} = \sup_{x,y\in\mathbb{Q}_p^n,\ x\neq y} \frac{|f(x) - f(y)|}{|x-y|_p^{\beta}}.$$

2.2 Auxiliary propositions and lemmas

In this part we state some auxiliary propositions and lemmas which will be needed for proving our main theorems. And we only describe partial results we need.

Firstly, the p-adic version of Hölder's inequality can be obtained in [8].

Lemma 2.1 (**Generalized Hölder's inequality on** $\mathbb{Q}_p^n$) Let $\mathbb{Q}_p^n$ be an n-dimensional p-adic vector space. Suppose that $q_1(\cdot), q_2(\cdot), r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$ and $r(\cdot)$ satisfy $\frac{1}{r(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$ almost everywhere. Then there exists a positive constant C such that for all $f \in L^{q_1(\cdot)}(\mathbb{Q}_p^n)$ and $g \in L^{q_2(\cdot)}(\mathbb{Q}_p^n)$, the inequality

$$\|fg\|_{L^{r(\cdot)}(\mathbb{Q}_p^n)} \leq C\|f\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \|g\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)},$$

holds.

In order to prove main theorem, the following norms of characteristic functions estimates are derived from [10], besides, the second part can be obtained by the following part (1) and plays a crucial role.

Lemma 2.2 (**Norms of characteristic functions**) If $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$. Then

$$\|\chi_{B_{\gamma}(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{\frac{\gamma n}{q(x,\gamma)}},$$

where

$$q(x,\gamma)=\begin{cases} q(x), & \text{if } \gamma<0, \\ q(\infty), & \text{if } \gamma\ge 0. \end{cases}$$

As we all known, the following Lemma give the boundedness of fractional integral operator on p-adic variable exponent Lebesgue space, for exact details, we can see [10].

Lemma 2.3 Assume $0<\alpha<n$, for all $r(\cdot),\ q(\cdot)\in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n),\ r_+<\frac{n}{\alpha}$ and $1/q(\cdot)=1/r(\cdot)-\alpha/n$, then

$$I_\alpha^p:\ L^{r(\cdot)}(\mathbb{Q}_p^n)\to L^{q(\cdot)}(\mathbb{Q}_p^n).$$

The following result is introduced in [12], it reveals that the fractional maximal operator can be controlled by the fractional integral operator.

Lemma 2.4 Let $0<\alpha<n$, for all $x\in\mathbb{Q}_p^n$, there exists a positive constant C, such that

$$|M_\alpha^p(f)(x)|\le C|I_\alpha^p(|f|)(x)|.$$

The authors in [16] obtained the following Lemmas 2.5, 2.6.

Lemma 2.5 Let $0<\beta<1,\ 0<\alpha<\alpha+\beta<n$. If $b\in\Lambda_\beta(\mathbb{Q}_p^n)$, then for any $x\in\mathbb{Q}_p^n$, we have

$$M_{\alpha,b}^p\le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}M_{\alpha+\beta}^p(f)(x).$$

Lemma 2.6 Let $0<\alpha<n$. If $b\in\Lambda_\beta(\mathbb{Q}_p^n)$ and $b\ge 0$, then for any $x\in\mathbb{Q}_p^n$ such that $M_\alpha^p(f)(x)<\infty$, we obtain

$$|[b,M_\alpha^p](f)(x)|\le M_{\alpha,b}^p(f)(x).$$

Finally, the following result is obtained by the definition 2.5, we also see the proof of theorem 3.4 in [15].

Lemma 2.7 Assume $\eta(\cdot)\in L^\infty(\mathbb{Q}_p^n),\ 1\le u<\infty,\ q(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n),\ \theta>0$ and $0\le\lambda<\infty$. If $f\in MK_{\lambda,q(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$, then, for all $l\in\mathbb{Z}$, there exists a constant $C>0$, such that

$$\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\le Cp^{l(\lambda-\eta(\cdot))}\|f\|_{MK_{\lambda,q(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)}.$$

By virtue of Lemma 2.2, we can deduce the following conclusion, which will simplify the proof of the main theorem.

Lemma 2.8 Let $0 < \alpha < n$, for all $q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$, with $q_1(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n),\ (q_1)_+ < n/\alpha$, and $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$, taking $k, l \in \mathbb{Z}$, then

$$\frac{1}{|B_k(x)|_h}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq \begin{cases} Cp^{\frac{(l-k)n}{q_1'(\cdot)}-k\alpha}, & \text{if } k < 0,\ l < 0, \\ Cp^{\frac{-kn}{q_1'(\infty)}+\frac{ln}{q_1'(\cdot)}-k\alpha}, & \text{if } k \geq 0,\ l < 0, \end{cases}$$

and

$$\frac{1}{|B_l(x)|_h}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq \begin{cases} Cp^{\frac{kn}{q_1(\cdot)}-\frac{ln}{q_2(\infty)}-l\alpha}, & \text{if } k < 0,\ l \geq 0, \\ Cp^{\frac{kn}{q_1(\infty)}-\frac{ln}{q_2(\infty)}-l\alpha}, & \text{if } k \geq 0,\ l \geq 0. \end{cases}$$

Proof: We divide the proof into four cases according to the range of $k,\ l$.

Case 1: If $k,\ l < 0$, for any fixed p-adic ball sphere $S_k,\ S_l \subset \mathbb{Q}_p^n$, using Lemma 2.2 and the fact $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$, we obtain

$$\frac{1}{|B_k(x)|_h}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{-kn}p^{\frac{kn}{q_2(\cdot)}}p^{\frac{ln}{q_1'(\cdot)}} = Cp^{\frac{(l-k)n}{q_1'(\cdot)}-k\alpha}.$$

Case 2: If $k < 0,\ l \geq 0$, similarly, using Lemma 2.2 and the fact $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$, we claim that

$$\frac{1}{|B_l(x)|_h}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{-ln}p^{\frac{kn}{q_2(\cdot)}}p^{\frac{ln}{q_1'(\infty)}} \leq Cp^{\frac{kn}{q_1(\cdot)}-\frac{ln}{q_2(\infty)}-l\alpha}.$$

Case 3: If $k \geq 0,\ l < 0$, for any fixed p-adic ball $B_k(x) \subset \mathbb{Q}_p^n$, using Lemma 2.2 and the fact $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$, we obtain

$$\frac{1}{|B_k(x)|_h}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{-kn}p^{\frac{kn}{q_2(\infty)}}p^{\frac{ln}{q_1'(\cdot)}} = Cp^{\frac{-kn}{q_1'(\infty)}+\frac{ln}{q_1'(\cdot)}-k\alpha}.$$

Case 4: If $k \geq 0,\ l \geq 0$, for any fixed p-adic ball $B_l(x) \subset \mathbb{Q}_p^n$, using Lemma 2.2 and the fact $1/q_2(\cdot) = 1/q_1(\cdot) - \alpha/n$, we obtain

$$\frac{1}{|B_l(x)|_h}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \leq Cp^{-ln}p^{\frac{kn}{q_2(\infty)}}p^{\frac{ln}{q_1'(\infty)}} \leq Cp^{\frac{kn}{q_1(\infty)}-\frac{ln}{q_2(\infty)}-l\alpha}.$$

Combining the cases 1, 2, 3, and 4, which implies the proof of Lemma 2.8.

3 Proofs of the principal results

Proof of Theorem 1.1 *If $f \in MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)$, we can write $f(z_1) = \sum_{l=-\infty}^{\infty} f(z_1)\chi_l(z_1)$. Suppose that k_0 is positive, since it is similar for $k_0 \leq 0$, using Minkowski's inequality, we obtain*

$$\|I_\alpha^p f\|_{MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)} = \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\|I_\alpha^p f\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$
\begin{aligned}
&\leq \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=-\infty}^{\infty}\|I_{\alpha}^{p}(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}^{u(1+\epsilon)}\right)\right)^{\frac{1}{u(1+\epsilon)}}\\
&\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=-\infty}^{k-2}\|I_{\alpha}^{p}(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&+ C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|I_{\alpha}^{p}(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&+ C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|I_{\alpha}^{p}(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&=: E_1+E_2+E_3.
\end{aligned}
$$

In response to E_2. On the one hand, if $k>0,\ l>0$, since $\eta(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$, by using definition 2.3, for $z_1\in S_k$, we have

$$
|\eta(z_1)-\eta(\infty)|\leq\frac{C}{\log_p(p+p^k)}\leq\frac{C}{k},
$$

which implies $p^{k\eta(z_1)}\approx p^{k\eta(\infty)}$, thus, the situation of $\eta(\infty)$ can be replaced by $\eta(z_1)$.

On the other hand, if $k<0,\ l<0$, since $\eta(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$, by using definition 2.3, for $z_1\in S_k$, we have

$$
|\eta(z_1)-\eta(0)|\leq\frac{C}{\log_p p^k}=\frac{C}{k},
$$

which implies $p^{k\eta(z_1)}\approx p^{k\eta(0)}$, thus, the situation of $\eta(0)$ can be replaced by $\eta(z_1)$.

Then by using Lemma 2.3, we obtain

$$
\begin{aligned}
E_2 &= C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|I_{\alpha}^{p}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=k-1}^{k+1}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\|f\chi_k\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&= C\|f\|_{MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

For any $k \in \mathbb{Z},\ l \le k-2$ and a.e. $z_1 \in S_k,\ z_2 \in S_l$, then $|z_1 - z_2|_p \approx p^k$, by Lemma 2.1, we obtain

$$|I^p_\alpha(f\chi_l)(z_1)| \le \int_{S_l} \frac{|f(z_2)|}{|z_1 - z_2|_p^{n-\alpha}} dz_2$$
$$\le Cp^{k(\alpha-n)} \int_{S_l} |f(z_2)| dz_2$$
$$\le Cp^{k(\alpha-n)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)}. \tag{3.1}$$

According to Minkowski's inequality, we obtain

$$E_1 \le C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=-\infty}^{k-2} \|I^p_\alpha(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}}$$
$$+ C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=0}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=-\infty}^{k-2} \|I^p_\alpha(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}}$$
$$=: E_{11} + E_{12}.$$

Then using Lemma 2.8 and (3.1)

$$E_{11} = C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=-\infty}^{k-2} \|I^p_\alpha f(\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}}$$
$$\le C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} p^{k(\alpha-n)} \right.\right.$$
$$\left.\left. \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}}.$$

For $k, l < 0$, we directly consider $q(\cdot)$ and the fact $\alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$. Let $\omega = \frac{n}{q_1'(\cdot)} - \eta(\cdot)$, then

$$E_{11} \le C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-2} p^{l\eta(\cdot)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} p^{\omega(l-k)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}}.$$

Now by using Fubini's theorem, Lemma 2.1 and the estimate $p^{-u(1+\epsilon)} < p^{-u}$, we have

$$E_{11} \le C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} [\epsilon^\theta \sum_{k=-\infty}^{-1} (\sum_{l=-\infty}^{k-2} p^{l\eta(\cdot)u(1+\epsilon)} \|f\chi_l\|^{u(1+\epsilon)}_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} p^{\frac{\omega(l-k)u(1+\epsilon)}{2}})$$

$$
\begin{aligned}
&\times \Big(\sum_{l=-\infty}^{k-2} p^{\frac{(l-k)\omega(u(1+\epsilon))'}{2}}\Big)^{\frac{u(1+\epsilon)}{(u(1+\epsilon))'}}\Big]^{\frac{1}{u(1+\epsilon)}}\\
&= C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\Big[\epsilon^{\theta}\sum_{k=-\infty}^{-1}\sum_{l=-\infty}^{k-2} p^{l\eta(\cdot)u(1+\epsilon)}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} p^{\frac{\omega(l-k)u(1+\epsilon)}{2}}\Big]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\Big[\epsilon^{\theta}\sum_{l=-\infty}^{-1} p^{l\eta(\cdot)u(1+\epsilon)}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\sum_{k=l+2}^{-1} p^{\frac{\omega(l-k)u}{2}}\Big]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\Big[\epsilon^{\theta}\sum_{l=-\infty}^{-1} p^{l\eta(\cdot)u(1+\epsilon)}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\Big]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\Big[\epsilon^{\theta}\sum_{l=-\infty}^{k_0} p^{l\eta(\cdot)u(1+\epsilon)}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\Big]^{\frac{1}{u(1+\epsilon)}}\\
&= C\|f\|_{MK^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Next, we need to estimate E_{12}, by applying Minkowski's inequality

$$
\begin{aligned}
E_{12} &\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}\|I_\alpha^p(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\quad + C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=0}^{k-2}\|I_\alpha^p(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\quad =: N_1 + N_2.
\end{aligned}
$$

Noting that the estimate N_2 is similar with the case of E_{11}, namely, by substituting $q_1'(\cdot)$ for $q_1'(\infty)$ and using the condition $\eta(\cdot) < n/q_1'(\infty)$.

As for N_1, since $\eta(\cdot) < \frac{n}{q_1'(\infty)}$, by using Lemma 2.8 and (3.1), we get

$$
\begin{aligned}
N_1 &\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}\|I_\alpha^p(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=-\infty}^{-1} p^{\frac{-kn}{q_1'(\infty)}} p^{\frac{ln}{q_1'(\cdot)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0} p^{(\eta(\cdot)-\frac{n}{q_1'(\infty)})ku(1+\epsilon)}\left(\sum_{l=-\infty}^{-1} p^{\frac{ln}{q_1'(\cdot)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}
\end{aligned}
$$

$$\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\left(\sum_{l=-\infty}^{-1} p^{\frac{ln}{q_1'(\cdot)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\left(\sum_{l=-\infty}^{-1} p^{\frac{ln}{q_1'(\cdot)}-l\eta(\cdot)}\|f(\chi_l)\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}p^{l\eta(\cdot)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}.$$

Combining the condition $\eta(\cdot)<\frac{n}{q_1'(\cdot)}$ *and Lemma 2,1, which implies that*

$$\begin{aligned}
N_1 &\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\left(\sum_{l=-\infty}^{-1} p^{\frac{ln}{q_1'(\cdot)}-l\eta(\cdot)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}p^{l\eta(\cdot)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{l=-\infty}^{-1} p^{l\eta(\cdot)(u(1+\epsilon))}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\times\left(\left(\sum_{l=-\infty}^{-1} p^{(\frac{ln}{q_1'(\cdot)}-l\eta(\cdot))(u(1+\epsilon))'}\right)^{\frac{u(1+\epsilon)}{(u(1+\epsilon))'}}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{l=-\infty}^{k_0} p^{l\eta(\cdot)(u(1+\epsilon))}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\leq C\|f\|_{MK^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

In order to estimate E_3, *let* $k\in\mathbb{Z}$ *and* $l\geq k+2$, *for a.e.*$z_1\in S_k,\ z_2\in S_l$, *then* $|z_1-z_2|_p\approx p^l$. *It follows from Lemma 2.1 that*

$$\begin{aligned}
|I^p_\alpha(f\chi_l)(z_1)| &\leq \int_{S_l}\frac{|f(z_2)|}{|z_1-z_2|_p^{n-\alpha}}dz_2\\
&\leq Cp^{l(\alpha-n)}\int_{S_l}|f(z_2)|dz_2\\
&\leq Cp^{l(\alpha-n)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \qquad (3.2)
\end{aligned}$$

Splitting E_3 *by using Minkowski's inequality we obtain*

$$\begin{aligned}
E_3 &= C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|I^p_\alpha(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\leq C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|I^p_\alpha(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}
\end{aligned}$$

$$+ C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=k+2}^{\infty}\|I_{\alpha}^{p}(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$=: E_{31} + E_{32}.$$

For E_{32}, taking $d = \frac{n}{q_1(\infty)} + \eta(\cdot) > 0$, by Lemma 2.8, (3.2), we have

$$\begin{aligned}
E_{32} &= C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}[\epsilon^{\theta}\sum_{k=0}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}(\sum_{l=k+2}^{\infty}\|I_{\alpha}^{p}(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)})^{u(1+\epsilon)}]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}[\epsilon^{\theta}\sum_{k=0}^{k_0} p^{\eta(\cdot)ku(1+\epsilon)}(\sum_{l=k+2}^{\infty}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}p^{l(\alpha-n)}\\
&\quad \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)})^{u(1+\epsilon)}]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}[\epsilon^{\theta}\sum_{k=0}^{k_0}(\sum_{l=k+2}^{\infty} p^{l(\alpha+\eta(\cdot))}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}p^{d(k-l)})^{u(1+\epsilon)}]^{\frac{1}{u(1+\epsilon)}}.
\end{aligned}$$

Similarly, we still make use of Lemma 2.1, definition 2.5 and Lemma 2.7 to obtain

$$\begin{aligned}
E_{32} &\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}[\epsilon^{\theta}\sum_{k=0}^{k_0}(\sum_{l=k+2}^{\infty} p^{l(\alpha+\eta(\cdot))u(1+\epsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}^{u(1+\epsilon)}p^{\frac{d(k-l)u(1+\epsilon)}{2}})\\
&\quad\times(\sum_{l=k+2}^{\infty} p^{\frac{(k-l)d(u(1+\epsilon))'}{2}})^{\frac{u(1+\epsilon)}{(u(1+\epsilon))'}}]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}[\epsilon^{\theta}\sum_{k=0}^{k_0}\sum_{l=k+2}^{\infty} p^{l\eta(\cdot)u(1+\epsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}^{u(1+\epsilon)}p^{\frac{d(k-l)u(1+\epsilon)}{2}}]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}[\sum_{k=0}^{k_0} p^{dku(1+\epsilon)/2}\sum_{l=k+2}^{\infty} p^{u(1+\epsilon)l(\lambda+\alpha-d/2)}]^{\frac{1}{u(1+\epsilon)}}\|f\|_{MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)}\\
&\le C\|f\|_{MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)}.
\end{aligned}$$

The last step is obtained by the fact $\lambda+\alpha<d/2$.

Now for E_{31}, again using Minkowski's inequality we obtain

$$\begin{aligned}
E_{31} &\le C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=k+2}^{-1}\|I_{\alpha}^{p}(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\quad + C\sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1} p^{\eta(\cdot)ku(1+\epsilon)}\left(\sum_{l=0}^{\infty}\|I_{\alpha}^{p}(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}
\end{aligned}$$

$$:= W_1 + W_2.$$

Noting that the estimate W_1 is similar with the case of E_{32}, namely, by substituting $q_1(\cdot)$ for $q_1(\infty)$ and using the condition $\frac{n}{q_1(\cdot)} + \eta(\cdot) > 0$, thus we omit the details.

In order to estimate W_2, by virtue to the fact $\lambda + \alpha - n/q_1(\cdot) < \eta(\cdot) < n/q_1'(\cdot)$, definition 2.5, Lemmas 2.7, 2.8, we have

$$\begin{aligned}
W_2 &= C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=0}^{\infty} \|I_\alpha^p(f\chi_l)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&\le C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=0}^{\infty} p^{l(\alpha-n)} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{Q}_p^n)} \right.\right. \\
&\quad \left.\left. \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&\le C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} p^{\eta(\cdot)ku(1+\epsilon)} \left(\sum_{l=0}^{\infty} p^{\frac{kn}{q_1(\cdot)}} p^{\frac{-ln}{q_2(\infty)}} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&= C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{-1} p^{(\eta(\cdot)+\frac{n}{q_1(\cdot)})ku(1+\epsilon)} \left(\sum_{l=0}^{\infty} p^{\frac{-ln}{q_2(\infty)}} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&\le C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \left(\sum_{l=0}^{\infty} p^{\frac{-ln}{q_2(\infty)}} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&\le C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \left(\sum_{l=0}^{\infty} p^{l\eta(\cdot)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)} p^{l(-\eta(\cdot)-\frac{n}{q_2(\infty)})} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&\le C \sup_{\substack{k_0\in\mathbb{Z}\\ \epsilon>0}} p^{-k_0\lambda} \left(\epsilon^\theta \left(\sum_{l=0}^{\infty} p^{l(\alpha-\eta(\cdot)-\frac{n}{q_1(\infty)}+\lambda)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \|f\|_{MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)} \\
&\le C\|f\|_{MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)}.
\end{aligned}$$

The last step is obtained by the fact $\lambda + \alpha - n/q_1(\infty) < \eta(\cdot)$.

In summary, combining the estimates of $E_1,\ E_2,\ E_3$, we can obtain

$$\|I_\alpha^p f\|_{MK_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)} \le C\|f\|_{MK_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{Q}_p^n)},$$

which implies the proof of theorem 1.1.

Proof of Theorem 1.2 *For all $x \in \mathbb{Q}_p^n$, by using Lemma 2.4, we obtain*

$$|M_\alpha^p(f)(x)| \le C|I_\alpha^p(|f|)(x)|.$$

Thus by using theorem 1.1, we can obtain the proof of theorem 1.2, thus we omit the details.

Proof of Theorem 1.3 *If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, then by using (1.1), the following elementary estimate*

$$\begin{aligned}
|[b, I_\alpha^p]f(x)| &\le \int_{\mathbb{Q}_p^n} \frac{|(b(x)-b(y))f(y)|}{|x-y|_p^{n-\alpha}} dy \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \int_{\mathbb{Q}_p^n} \frac{|f(y)|}{|x-y|_p^{n-\alpha-\beta}} dy \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} |I_{\alpha+\beta}^p f(x)|,
\end{aligned}$$

holds under the assumptions of theorem 1.3, by using theorem 1.1, we directly obtain the result of theorem 1.3, thus we omit the details.

Proof of Theorem 1.4 *For any fixed $x \in \mathbb{Q}_p^n$, if $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, then by Lemma 2.5, we have*

$$M_{\alpha,b}^p \le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M_{\alpha+\beta}^p(f)(x).$$

Thus by using theorem 1.2, we can obtain the proof of theorem 1.4, thus we omit the details.

Proof of Theorem 1.5 *For any fixed $x \in \mathbb{Q}_p^n$, if $b \in \Lambda_\beta(\mathbb{Q}_p^n)$* and $b \ge 0$, *then by Lemma 2.6, we have*

$$|[b, M_\alpha^p](f)(x)| \le M_{\alpha,b}^p(f)(x).$$

Thus by using theorem 1.4, we can obtain the proof of theorem 1.5, thus we omit the details.

Proof of Theorem 1.6 *For any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, by applying triangle inequality, we obtain*

$$\begin{aligned}
|[|b|, M_p^\sharp]f(x)| &= |\sup_{\gamma\in\mathbb{Z}} \frac{|b(x)|}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|dy \\
&- \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} ||b(y)|f(y) - (|b|f)_{B_\gamma(x)}|dy| \\
&\le \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} ||b(x)| - |b(y)|||f(y)| + |b(x)f_{B_\gamma(x)} - (bf)_{B_\gamma(x)}|dy \\
&\le \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} ||b(x)| - |b(y)|||f(y)|dy \\
&+ \sup_{\gamma\in\mathbb{Z}} |\frac{|b(x)|}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} f(z)dz - \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(z)|f(z)dz|
\end{aligned}$$

$$\le 2M^p_{|b|}f(x).$$

By using $M^\sharp_p(f) \le 2M^p(f)$, *if* $x \in \mathbb{Q}^n_p$, *then*

$$\begin{aligned}|[b, M^\sharp_p](f)(x)| &\le 2((b^-(x))M^\sharp_p(f)(x) + M^\sharp_p(b^- f)(x)) + |[|b|, M^\sharp_p](f)(x)| \\ &\le 4((b^-(x))M^p(f)(x) + M^p(b^- f)(x)) + 2M^p_{|b|}f(x).\end{aligned}$$

In view of $b \ge 0$ and $b \in \Lambda_\beta(\mathbb{Q}^n_p)$, *we only make use of the result of Corollary 1.17, it is easy to obtain the proof of theorem 1.6.*

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Conflict of interest

The authors state that there is no conflict of interest.

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Author contributions

All authors contributed equally to the writing of this article. All authors read the final manuscript and approved its submission.

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